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A Non-Wiener Random Walk in a Two-Dimensional Bernoulli Environment

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For a degenerate random walk in a 2D Bernoulli environment without local traps, computer results show a non-Wiener behavior. For a better exploitation of the memory, the analysis is based on the statistics of the first exit time from a square.

KEY WORDS: Random walks in random environments; testing normality and independence; Erdős–Kac statistics; Monte Carlo methods.

1. INTRODUCTION

There are delicate problems where a conjectured anomaly differs from the regular behavior very little, e.g., a logarithmic correction versus a power of time. In these cases even the use of powerful computers requires increased care because of limitations in memory and computing time.

An example is the asymptotic behavior of random walks in a random environment (RWiRE). Here we suppose that the environment is Bernoulli, i.e., it is translation-invariant and, for different sites, the random transition probabilities are independent. [In the Bernoulli case, for $d \geqslant 2$, the typical behavior is meant to be (strictly) diffusive, i.e., one with an asymptotically linear increase of the mean square displacement and with a Wiener limiting process for trajectories in the standard diffusion scaling. For $d = 1$, Sinai[(10)] showed that the displacement of a n.n. RWiRE z_n: $n \in \mathbb{Z}_+$ satisfying a suitable centralization condition increases like $(\log n)^2$, i.e., the behavior is strongly subdiffusive.]

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According to general belief, there are two critical dimensions $d_{cr} \leqslant \bar{d}_{cr}$ determined as follows: if $d \geqslant \bar{d}_{cr}$, then any nondegenerate RWiRE on $\mathbb{Z}^d$ satisfying a suitable centralization condition is diffusive ("nondegenerate" means that the random transition probabilities are uniformly bounded away from 0) and $\bar{d}_{cr}$ is the smallest integer with this property; if $d \geqslant d_{cr}$, then sufficiently small random perturbations of the simple symmetric random walk (SSRW) satisfying a suitable centralization condition behave diffusively and d_{cr} is the smallest integer with this property [the SSRW is determined by the nonrandom transition probabilities $p(x, y) = (2d)^{-1}$ if $x, y \in \mathbb{Z}^d$, $|x - y| = 1$, while a perturbation is considered small if, for a suitable $\varepsilon = \varepsilon(d) > 0$, $\operatorname{Prob}(|p(x, y) - (2d)^{-1}| \leqslant \varepsilon) = 1$ for any $x, y \in \mathbb{Z}^d$, $|x - y| = 1$].

We remark that even the formulation of the suitable centralization condition is quite a hard problem.

Relying upon formal perturbation theory and renorm-group methods, several authors have agreed that $d_{cr} \sim 2$,[3,5,7,9] i.e., in case of small disorder, for $d = 2 + \varepsilon$ the behavior is diffusive, while for $d = 2 - \varepsilon$ it is subdiffusive (in fact, Fisher[3] showed that $\langle z^2(t) \rangle \sim t^{1-\varepsilon^2}$). As to the critical case $d = 2$, the results also agree that in the increase of the mean square displacement there may be at most a logarithmic correction to the linear one. Fisher[3] obtains $\langle z^2(t) \rangle \asymp t(1 + c/\log t)$ and Obukhov[7] even derives a strictly diffusive behavior. Fisher[3] also gives intuitive arguments for the extension of the aforementioned methods to the large-disorder case. On the other hand, Bramson and Durrett[1] outline a rigorous construction for nondegenerate RWiRE on any $\mathbb{Z}^d$, $d \geqslant 1$, where the nondegenerate environment is translationally invariant but not a Bernoulli one and the behavior is subdiffusive. Since the environments of their example obey an exponential mixing condition, they stress that similar phenomena may also occur for Bernoulli environments, thus casting doubt on the folk belief that $d_{cr} < \infty$. The environment they construct is given by a random potential containing larger and larger traps visited by the random walker sufficiently often and this slowdown produces the subdiffusivity.

Earlier, Marinari *et al.*[6] made numerical experiments in order to check whether the behavior of a two-dimensional random walk in a Bernoulli environment had a subdiffusive character. The model studied by them looks as follows. To each site in $\mathbb{Z}^2$ there are assigned four numbers Q_i ($i = 1, 2, 3, 4$) uniformly distributed in $(0, 1)$. The transition probability to each of the four neighboring sites is defined by $p_i(k) = Q_i^k (\sum_{j=1}^4 Q_j^k)^{-1}$, where k is a parameter ranging between 0 and $+\infty$. For $k = 0$ this model reduces to the SSRW, while in the limit $k \to \infty$ only one edge is likely. The slowdown of the random walk with increasing k is shown in Table 3 of Ref. 6. It is clear that trapping configurations arise in the limit $k \to \infty$.

Obukhov[7] interpreted the observations of Marinari *et al.* as follows: He calculated the distribution function of the exit time from the simplest quasitrap (two neighboring sites) for finite k. He proved that above a critical k the expected exit time is infinite (in the two-dimensional case the critical k value is equal to 6). He concluded that the anomalous effects are caused by these local quasitraps, which, of course, do not occur in a nondegenerate model.

An intriguing question is really whether traps analogous to the ones of Ref. 1 can be localized sufficiently often in a Bernoulli environment (it is worth stressing that we do not understand exactly any of the terms "trap," "analogous," "localized," "sufficiently often" of the last clause, but a clever simulation may help to imagine the geometry). Our aim, however, is more modest: we want to decide whether non-Wiener behavior can occur at all in a Bernoulli environment without local traps. To get a more characteristic answer, we even drop the nondegeneracy condition.

Our simple model is motivated by the situation described above. We have studied the "complementary" case to the limit $k = \infty$: for each site there are three possible edges, while the fourth one is forbidden. More formally: let f be a random variable assigned independently to each site with $P(f = e_i) = 1/4$ $(i = 1,..., 4)$, where e_i $(i = 1,..., 4)$ are the unit vectors in $\mathbb{Z}^2$ and the transition probability to each of the four neighboring sites is

$$p_i = \begin{cases} 0 & \text{if } e_i = f \\ 1/2 & \text{if } e_i = -f \\ 1/4 & \text{otherwise} \end{cases}$$

An elementary combinatorial consideration shows that in our model there are no finite (local) trapping configurations. It is obvious (see Fig. 1) that the boundary of the convex hull of any possible finite trap should contain at least one site x through which there exists a supporting line of the convex hull not parallel with the coordinate axes. Now there are two edges at x showing outside from the convex hull and both of them cannot be forbidden. On the other hand, arbitrarily large forbidden regions, sort of scatterers (see Fig. 2), can exist. The configuration of Fig. 3 can be considered as an almost trapping one. The probability of such a configuration is proportional to exp[perimeter of the area $\cdot \log(1/4)$]. So the expected exit time is finite.

In Section 2 we briefly describe the statistical method used in our evaluation of the computer results. The essential argument for using statistics based upon the first exit time from a square is that this ensures a maximal exploitation of the piece of the environment saved in the memory. Section 3 summarizes the numerical results, while Section 4 draws the conclusions.

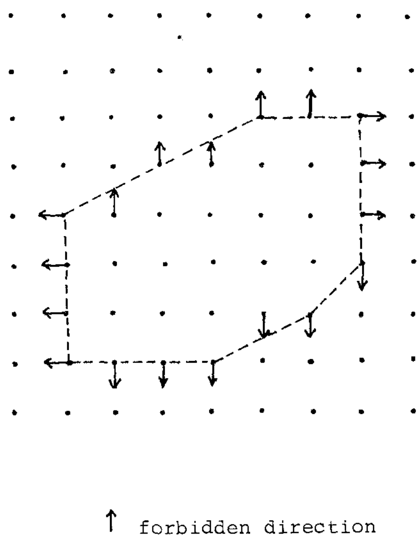

Fig. 1. Nonexistence of finite traps.

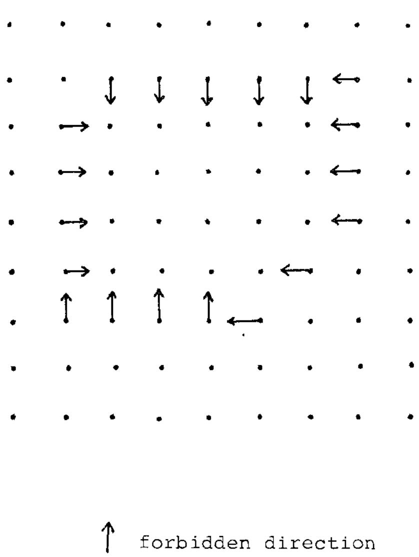

Fig. 2. Example of a finite scatterer.

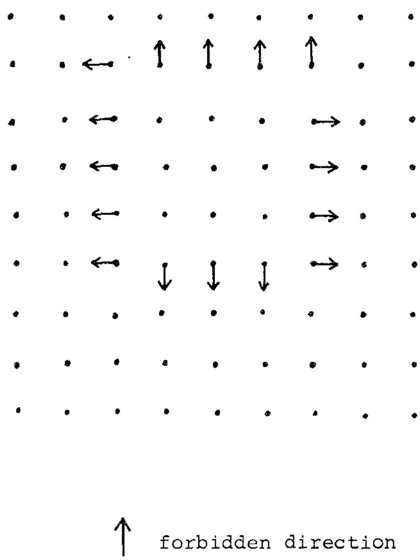

Fig. 3. Example of an almost trap.

2. STATISTICAL CONSIDERATION

Our task is to test the hypothesis H_0 that the normed trajectory of the moving particle $A^{-1/2}z(At)$ obeys the functional central limit theorem. Our statistical considerations follow the standard textbook concepts (e.g., Ref. 4). There are different standard methods for testing the normality of $t^{-1/2}z(t)$: e.g., methods of semiinvariants, χ^2 square test, Kolmogorov–Smirnov test, Sarkadi test. The above methods require numerical experiments generating trajectories, which leave the given random environment (region) in the course of the observation with a small but not negligible probability.

For a region of a given size and a fixed time necessary for the standard tests, the typical trajectories leave the region earlier than the admissible observation time for the standard statistical methods. In order to exploit the information supplied by the typical trajectories of the moving particle until leaving the region, we propose the statistical investigation of the distribution of the exit time. Let x and y denote the two coordinates of z. Set

$$ {}_xT_a = \min\{k \mid k \geqslant 1, x(k) \in \mathbb{Z}\setminus[-a, a], x(0) = 0\} $$

It is well known that for a symmetric random walk with diffusion coefficient σ on the lattice $\mathbb{Z}$ the following result of Erdős and Kac[2] is true: the exit time of the scaled random walk is asymptotically equal to the exit time of the Wiener process, i.e.,

$$\lim_{N \to \infty} P_0 \left[T_N > t \frac{N^2}{\sigma^2} \right] = 1 - F(t), \qquad t \geqslant 0 \tag{$*$}$$

where

$$F(t) = 1 - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left[-\frac{\pi^2}{8} (2k+1)^2 t \right]$$

For a two-dimensional SSRW the exit time $T_{a,b}$ from the region $[-a, a] \times [-b, b]$ under the condition that $z(0) = 0 \in \mathbb{Z}^2$ is by definition ${}_zT_{a,b} = \min\{{}_xT_a, {}_yT_b\}$, where $x(t)$ and $y(t)$ are the two coordinates of $z(t)$. In the SSRW case the ${}_xT_a$ and ${}_yT_b$ are asymptotically independent. So ${}_zT_{a,b}$ can be regarded as the minimum of two independent random variables with distributions $F(\sigma^2 t/a^2)$ and $F(\sigma^2 t/b^2)$, respectively, where $\sigma^2 = 1/2$.

3. NUMERICAL RESULTS

The hypothesis that $A^{-1/2} z(At)$ $(A \to \infty)$ is asymptotically a 2D Wiener process and the invariance of its 2×2 covariance matrix under the rotation by $\pi/2$ involve that the x and y components are asymptotically independent Wiener processes with a common unknown variance σ^2. Thus, we should test the hypothesis

$$\begin{aligned} H_0\colon \quad \{ {}_zT_{a,a} = \min\{{}_xT_a, {}_yT_a\} &\text{ has a distribution function of the form} \\ G_{\sigma^2}(t)\colon \quad = 1 - [1 - F(\sigma^2 t/a^2)]^2 &\text{ for some } \sigma^2 \} \end{aligned}$$

Notice that H_0 is a composite hypothesis, allowing σ^2 to vary in $(0, \infty)$.

Having a large sample (our simulation considers 50 independent random environments of size 401×401 with 200 random walks in each of them), there are several possibilities to estimate the parameter σ^2 of the distribution function $G_{\sigma^2}(t)$. One of them is the "minimum χ^2 estimate": for a given partition of the t axis, say $(-\infty, t_1]$, $(t_1, t_2]$,..., (t_k, ∞) $(k > 1)$, one looks for a value σ^2 for which the χ^2 distance between the empirical probabilities

$$P_i = P\{{}_zT_{200,200} \in (t_i, t_{i+1}]\} \quad (t_0 = -\infty, t_{k+1} = +\infty)$$

and the theoretical probabilities $Q_i = G_{\sigma^2}(t_{i+1}) - G_{\sigma^2}(t_i)$ is minimal. Another way is to minimize the J divergence $\sum_i (P_i - Q_i) \log(P_i/Q_i)$ between the aforementioned two distributions.

Table I[a]

σ^2	χ^2	σ^2	χ^2
0.10	9780635	0.44	271.30
0.15	215139.5	0.45	389.75
0.20	29176.89	0.50	1290.22
0.25	7564.57	0.55	2695.83
0.30	2137.00	0.60	4652.31
0.35	399.96	0.65	7262.09
0.36	247.53	0.70	10680.94
0.37	139.03	0.75	15124.88
0.38	69.05	0.80	20883.84
0.39	33.35	0.85	28341.40
0.40	28.51	0.90	38003.29
0.41	51.84	0.95	50533.43
0.42	101.15	1.00	66804.00
0.43	174.76		

[a] Square: 401×401. df $= 9$.

First we see how strongly the estimate depends on the chosen distance and partition. For both distances and all reasonable partitions we obtain $\sigma^2 \in [0.39, 0.40]$. As an example, Table I shows the dependence of the χ^2 distance on σ^2 for the partition consisting of ten cells equiprobable with respect to $G_{0.5}(t)$.

For testing the hypothesis H_0 we used χ^2 statistics with two partitions consisting of 10 and 33 cells equiprobable with respect to $G_{0.4}(t)$. Our conclusion is that for no value σ^2 is the shape of the empirical exit time distribution acceptable as $G_{\sigma^2}(t)$. Tables II and III show the behavior of χ^2 around its minimum for 10 and 33 cells partitions, respectively.

Table II[a]

σ^2	χ^2	σ^2	χ^2
0.385	52.856	0.395	41.955
0.386	50.342	0.396	42.557
0.387	48.152	0.397	43.458
0.388	46.283	0.398	44.655
0.389	44.732	0.399	46.146
0.390	43.496	0.400	47.929
0.391	42.572	0.401	50.002
0.392	41.959	0.402	52.363
0.393	41.653	0.403	55.010
0.394	41.652	0.404	57.941

[a] Square: 401×401. df $= 9$.

Table III[a]

σ^2	χ^2	σ^2	χ^2
0.385	93.720	0.395	86.677
0.386	91.426	0.396	87.867
0.387	89.492	0.397	89.391
0.388	87.921	0.398	91.242
0.389	86.700	0.399	93.428
0.390	85.827	0.400	95.940
0.391	85.313	0.401	98.775
0.392	85.137	0.402	101.932
0.393	85.311	0.403	105.418
0.394	85.823	0.404	109.216

[a] Square: 401×401. df $= 32$.

Table IV[a]

σ^2	χ^2	σ^2	χ^2
0.390	105.314	0.395	102.802
0.391	104.151	0.396	103.277
0.392	103.321	0.397	104.076
0.393	102.820	0.398	105.191
0.394	102.649	0.399	106.623

[a] Square: 201×201. df $= 9$.

Table V[a]

σ^2	χ^2	σ^2	χ^2
0.387	178.354	0.392	175.174
0.388	177.958	0.393	175.684
0.389	175.930	0.394	176.554
0.390	175.303	0.395	177.803
0.391	175.050	0.396	179.413

[a] Square: 201×201. df $= 32$.

Table VI[a]

σ^2	χ^2	σ^2	χ^2
0.388	50.296	0.393	48.447
0.389	49.294	0.394	49.016
0.390	48.611	0.395	49.891
0.391	48.243	0.396	51.071
0.392	48.188	0.397	52.555

[a] Square: 301×301. df = 9.

Though, of course, our tables only give a discretization of the function $\chi^2(\sigma^2)$, the smoothness of this function implies that the conclusion is true at any reasonable signifinance level (for 9 and 32 degrees of freedom the critical χ^2 values at level 99.95% are 29.666 and 64.995, respectively).

The same statistics were computed for 201×201 and 301×301 square (parts of the same environments as for the 401×401 case). Tables IV–VII show the behavior of the χ^2 values around their minima for these sizes, resulting the same conclusion as before.

Remark. The invariance of the covariance matrix of $A^{-1/2}z(At)$ under the rotation by $\pi/2$ implies that even in the non-Wiener case the components are uncorrelated. Nevertheless, it may happen that the two components are asymptotically dependent. For the sake of completeness we tested the independence of the exit times ${}_xT_a$ and ${}_yT_b$ for a given 101×101 environment based on 1000 trajectories, again using the χ^2 test with 5×5 cells (4×4 degrees of freedom). For cells equiprobable with respect to $F(0.5t/50^2)$ we find $\chi^2 = 11.64$. This value is between the 10% and 90% quantiles of χ^2_{16} (9.312 and 23.542). Consequently, this value is consistent with the asymptotic independence of the components.

Table VII[a]

σ^2	χ^2	σ^2	χ^2
0.388	110.469	0.393	108.205
0.389	109.287	0.394	108.828
0.390	108.479	0.395	109.788
0.491	108.036	0.396	111.103
0.392	107.942	0.397	112.760

[a] Square: 301×301. df = 32.

4. CONCLUSIONS

1. For the RWiRE introduced in Section 1, statistical evaluations based upon the first exit time of the random walk from a square show that asymptotically this RWiRE is not a Wiener process.

2. The hypothesis that the axial components of the random walk are independent can be accepted on the basis of our samples.

3. The RWiRE considered is degenerate, but we expect that, by applying our numerical and statistical methods, the diffusivity of nondegenerate RWiREs can also be checked.

4. At present, we are unable to use our numerical method for suggesting a rigorous proof for the nondiffusivity of this or other Bernoulli RWiREs.

5. Our method should also be useful for checking the diffusivity of other processes where memory constraints may arise, e.g., to check whether, on R^1, the trajectory of a Brownian point particle of fixed mass M interacting with an ideal gas of identical point particles of mass 1 through elastic collisions is asymptotically Wiener or not (cf. Refs. 8 and 11).

We remark that recent numerical results by Sinai's group (personal communication) suggest a nondiffusive behavior.

5. TECHNICAL REMARKS

We have used the RNDM2 random number generator from the CERN LIBRARY. On the IBM 3031 under OSVS1 the required time for the simulation was 20 hr CPU. The length of a trajectory was bounded by 200,000 steps. For every sample the time and site of the exit from the 401×401 square are saved, so future proposed statistics can be computed. The reliability of the numerical results obtained was checked by also carrying out all methods for the SSRW. The results are in a good agreement with the theory, e.g., the empirical distribution of the exit time fits the theoretical one: the χ^2 value for 9 degrees of freedom was 8.456, which is inside the interval between the 40% and 60% quantiles.

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